Measure Theory with Ergodic Horizons
Lecture 14

Examples of Haar measures on loc. compact groups.

(a) Any ctbl group (with discrete topology) with the counting measure.
(b) $(\mathbb{R}^n, +)$ with Lebesgue measure
(c) $(\mathbb{R}_{>0}, \cdot)$ with the pushforward of Lebesgue measure under $x \mapsto e^x$.
(d) $(S^1, \cdot)$ viewed as a subgroup of $(\mathbb{C} \setminus \{0\}, \cdot)$. This is a compact group and the Haar probability measure on it is the pushforward of Lebesgue measure from $[0,1)$ under $x \mapsto e^{2\pi x i}$.
(e) $\left(\prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}\right.$, coordinatewise addition). This is isomorphic to $2^{\mathbb{N}}$ and the Haar probability measure is the Bernoulli$(\frac{1}{2})$ measure since it is invariant under bit-flips (changing any bit from 0 to 1 and vice versa).
(f) $GL_n(\mathbb{R}) :=$ the group of invertible $n \times n$ matrices under matrix multiplication. We can view $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n \times n} \cong \mathbb{R}^{(n^2)}$ and as such it is a closed subset because a matrix $A \in GL_n(\mathbb{R}) \iff \det(A) \neq 0$, and det is a continuous function on $\mathbb{R}^{(n^2)}$. One can show that $\{A \in \mathbb{R}^{(n^2)} : \det(A) = 0\}$ has empty interior and moreover is Lebesgue null, hence $GL_n(\mathbb{R})$ is a conull subset of $\mathbb{R}^{(n^2)}$ but the Haar measure on $GL_n(\mathbb{R})$ isn't the restriction of the Lebesgue measure.

Note that we have proven ergodicity of certain actions on those groups, but they are all of the following form:

Theorem. Let G be a loc. compact Hausdorff group equipped with a Haar measure μ. Then the translation action of a ctbl dense subgroup $\Gamma \leq G$ is ergodic.

Proof. Like with special cases the proof follows from the 99% with open sets. $\square$

Borel / measure isomorphism theorems.

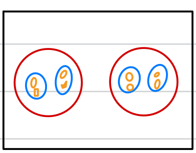
The following is one of the basic theorems of descriptive set theory which is used casually in other subjects like probability and dynamics:

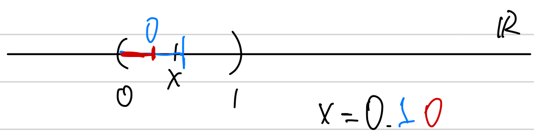
Borel isomorphism thm. Any two unctbl Polish spaces X, Y are Borel isomorphic, i.e. $\exists$ Borel isomorphism $f: X \to Y$ (i.e. f is a bijection and f and f^{-1} are Borel). In particular, all unctbl Polish spaces are Borel isomorphic to $2^{\mathbb{N}}$ (or $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$), and have cardinality continuum.

The proof of this theorem following two theorems via the Borel version of the Cantor–Schröder–Bernstein theorem:

Theorem. Let X be a Polish space.
(a) Cantor-Bendixson: $2^{\mathbb{N}} \hookrightarrow X$ continuously embeds, if X is unctbl.
(b) There is a Borel injection $X \hookrightarrow 2^{\mathbb{N}}$, which I call a binary representation.

Proof-sketch of (a). Cantor-Bendixson says that every Polish space X admits a unique partition $X = P \sqcup U$, where U is a ctbl open set U and a closed perfect set P (perfect := no isolated points). Because P is closed, it's a Polish space and Cantor's Perfect Set theorem says that $2^{\mathbb{N}}$ embeds into every nonempty perfect Polish space. Thus X is unctbl $\Leftrightarrow$ P is nonempty $\Leftrightarrow$ $2^{\mathbb{N}} \hookrightarrow P$. $\square$





Proof of (b). Fix a ctbl basis $(U_n)_{n\in\mathbb{N}}$ for X. The map $c: X \to 2^{\mathbb{N}}$ defined by $x \mapsto c(x)$ where $c(x)(n) := \begin{cases} 1 & x \in U_n \\ 0 & x \notin U_n \end{cases}$. This is injective due to Hausdorffness of X, i.e. the fact that for any distinct $x_0, x_1 \in X$ $\exists\, U_n$ with $x_0 \in U_n$ and $x_1 \notin U_n$. This is a Borel function because $c^{-1}([**\underset{n}{0}***]) = U_n^c$ and $c^{-1}([**\underset{n}{1}***]) = U_n$, so the preimage of every cylinder $[w]$ is a finite intersection of closed and open sets. $\square$

Def. A measurable space $(X, \mathcal{I})$ is called *standard* if there is a Polish metric on X so that $\mathcal{I} = \mathcal{B}\text{orel}(X)$. In other words, X was a Polish space but we forgot the topology and only kept the Borel σ-algebra.

Remark. The Borel isomorphism theorem states that for an unctbl X, there is only one, up to isomorphism, standard σ-algebra.

Def. For measure spaces $(X, \mathcal{I}, \mu)$ and $(Y, \mathcal{J}, \nu)$, a function $f: X \to Y$ is called *measure isomorphism* if there are conull sets $X' \subseteq X$ and $Y' \subseteq Y$ such that $f|_{X'}: X' \to Y'$ is a bijection and $f|_{X'}$ and $(f|_{X'})^{-1}$ are $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{J}, \mathcal{I})$ measurable, and $f_* \mu = \nu$.

Def. A *standard measure space* is a measure space $(X, \mathcal{I}, \mu)$ where $(X, \mathcal{I})$ is a standard Borel space and μ is a σ-finite measure on $\mathcal{I} = \mathcal{B}(X)$.

Measure Isomorphism Theorem. Any atomless standard probability space $(X, \mathcal{B}, \mu)$ is measure isomorphic to $([0,1], \lambda)$. In fact, there is a Borel isomorphism $f: X \to [0,1]$, such that $f_* \mu = \lambda$.

Proof. By the Borel isomorphism theorem, there is a Borel isomorphism $g: X \to [0,1]$. Thus, replacing X with $[0,1]$ and μ with $g_* \mu$, we may assume without loss of generality, that $X = [0,1]$ and μ is an atomless Borel probability measure on $[0,1]$. Thus, it remains to prove that there is a Borel isomorphism $f: [0,1] \to [0,1]$ such that $f_* \mu = \lambda$. This will be done next time by analyzing all locally finite Borel measures on $\mathbb{R}$. $\boxtimes$

Cor. Every standard infinite atomless measure space is measure isomorphic to $(\mathbb{R}, \lambda)$.

Proof. HW.